The Bernoulli Hump Generated by a Submarine

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Introduction

Submarines have the advantage that they are covert. Nevertheless they can produce various effects at the sea surface and, though these may be extremely small, they offer a potential for detecting the vessel. The purpose of this paper is to re-examine earlier work on the Bernoulli hump. Some of the original work appears to be due to Hershey [1]. This includes a description of the theory and a comparison with model tests. The model is based on the Rankine ovoid, which is also called a “doublet” and consists of a single source and sink. The methodology is employed by Stefanick [2] to prepare tables of the amplitudes of the Bernoulli hump created by various submarine types and of the amplitudes of the Kelvin wake along the submarine tracks. These tables are reproduced by Daly [3] in a subsequent study on submarine detectability.

In this study, more details of the theory are provided and the scope of the modeling is broadened.

The Hull Model

Unlike submarines from WW2, which spent most of their time on the surface, today’s submarines need to surface much less frequently and their shape can be optimized for sub-surface speed and endurance. Wave-making resistance is usually not thought to be an issue so that a pointed bow is not necessary.

The hull can be represented as a distribution of sources and sinks so that the fluid flow is close to that over a real hull. Because of approxmate axial symmetry over each transverse section of the main hull, we can employ a simple distribution over its longitudinal axis. This is a sub-class of the “thin ship” approximation. At any longitudinal position, the sources and sinks forward of that position must introduce or absorb fluid in such a way that the streamlines in the absence of the hull resemble the streamlines when the hull is present.

The simplest hull model representing a submerged submarine is a doublet, which consists of a point source of fluid near the bow and an equal sink at the same depth near the stern. Because a vessel does not normally introduce additional fluid into the medium, the source and sink strengths must cancel. The source strength is chosen so that one of the streamlines corresponds to the actual submarine hull. Apart from extreme simplicity, the doublet has the advantage that, at sufficient distances, the flow closely approximates that of any hull. However, at distances of the order of the submarine length or less, the approximation may be poor.

The source strength, S, can be related to the flow through the submarine cross section; assuming a cylindrical hull of maximum radius, a, usually at or near mid-ships and adopting the thin ship approximation, we have:

$$S = \pi a^2 U_0 , \quad (1)$$

where U_0 is the submarine speed. The source and sink are typically separated by a distance, d, of the order of the submarine length, L.

The positions of the sources should be chosen so that the flow at the bow and stern are correctly located. At the bow, this occurs approximately when the forward flow from the source just cancels the undisturbed flow speed and yields a stagnation point at the extreme forward position. For the doublet a similar situation pertains at the stern. It is soon verified that the source should be positioned on the axis, at an approximate distance $a/2$ behind the most forward position of the hull. The sink at the stern should be placed forward of the extreme stern position by the same distance, i.e.

$$L = d + a . \quad (2)$$

Figure 1 shows the streamline corresponding to (half) the doublet hull for a submarine of length 100 m and beam 10 m (a = 5 m). It is symmetrical across a transverse plane at mid-ships. The calculation simply involves the vectorial addition of the free stream velocity to the velocities from the source and sink. The hull is blunt at both bow and stern and the mid-ships section is long and almost uniform. This shape corresponds quite well with the hulls of many real submarines except for the stern, which is likely to result in excessive turbulence due to boundary layer separation.

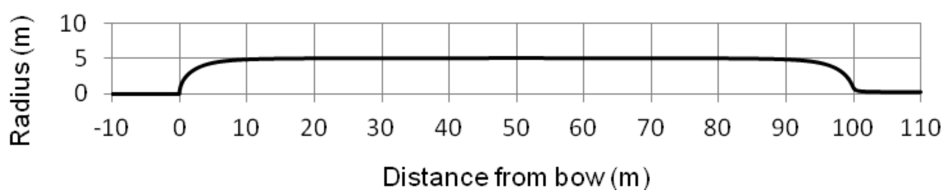

Figure 1. Streamline corresponding to the hull represented by a doublet.

To minimize viscous drag, modern submarine hulls have a tear drop shape. The bow is blunt and the hull tapers off towards the stern. The shape usually exhibits approximate cylindrical symmetry and resembles a three dimensional version of an aircraft wing. The design is very streamlined and one of the effects is that boundary layer separation and the effects of turbulence are reduced. This results in a low acoustic signature as well as low drag. (However, this shape is far from optimal if wakes of gravity waves are important. The Kelvin wake and internal wave wakes fall into this category.)

The first submarine to adopt a tear drop hull was USS Albacore. Figure 2 shows a profile of a hull model with this shape. (As before, the sail has been omitted.) It is cylindrically symmetric about the longitudinal axis. The bow comprises a hemisphere attached to a tapered section that consists of a parabola of revolution. We call this part of the model a Wigley hull, because its derivation is similar to that of the Wigley hull familiar from surface ship modeling. If the overall length is L and the radius of the bow section is a, which is equal to half of the submarine beam as well as half the submerged draft, the length of the Wigley section is $L - a$. Denoting the distance from the bow as x, the radius, r, of the Wigley section can be expressed as:

$$r = a\left(1 - \frac{(x-a)^2}{(L-a)^2}\right) \tag{3}$$

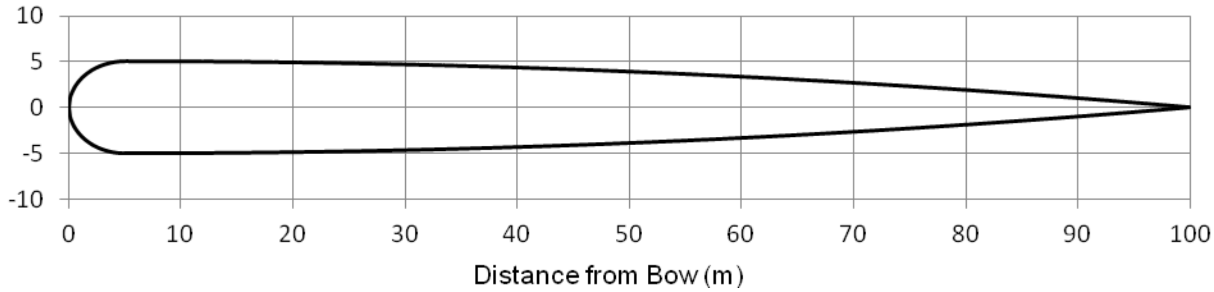

Figure 2. Model submarine hull.

At any longitudinal position, the sources and sinks forward of that position must introduce fluid so that the streamlines in the absence of the hull resemble the streamlines when the hull is present. Therefore, if the speed is U_0, we have:

$$\int \sigma dx = \pi r^2 U_0 \,, \tag{4}$$

where σ is the source density along the central submarine axis. Differentiating yields for the Wigley part:

$$\sigma = 2\pi r U_0 \frac{dr}{dx} = -4\pi a^2 U_0 \frac{x-a}{(L-a)^2}\left(1 - \frac{(x-a)^2}{(L-a)^2}\right). \tag{5}$$

As with the doublet, the hemispherical bow section can be represented by a single source. This is situated at a distance of $a/2$ from the extreme forward position. The dimensions of σ and S are m^2/s and m^3/s, respectively.

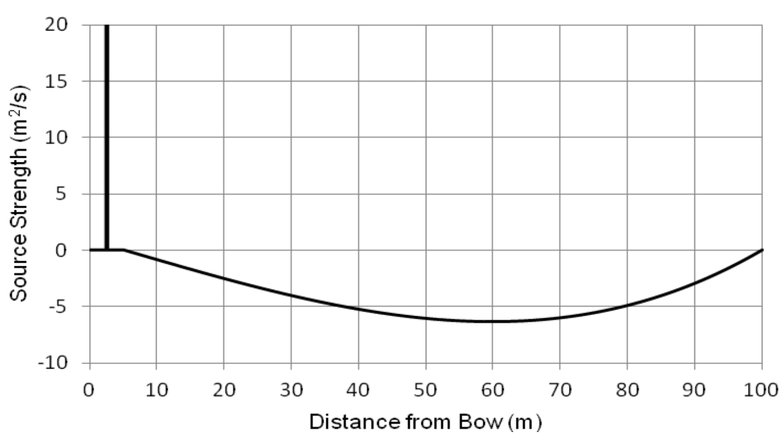

Figure 3. The source distribution for the model hull of length 100 m and beam 10 m; the speed is 5 m/s. The area under the bow spike is about 392.7 m^3/s.

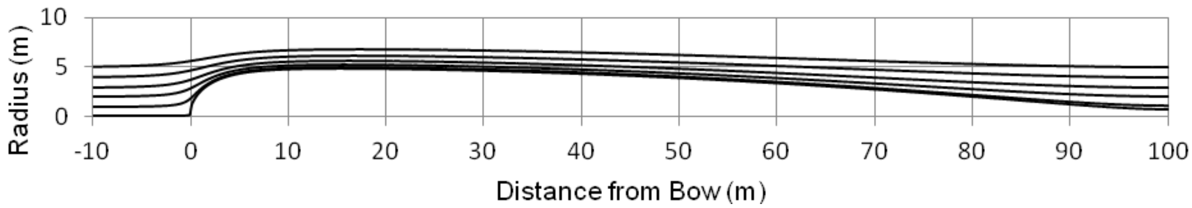

Figure 4. Streamlines calculated from the source distribution. (U_0 = 5 m/s.)

Figure 3 shows the source distribution for the model. This exhibits a large spike associated with the blunt bow followed by a smoothly varying distribution corresponding to the Wigley section. It has been verified that the integral over the source distribution is zero, which is required because the submarine does not introduce any fluid.

The representation is approximate but we can determine if there are discrepancies by plotting a few streamlines derived from the distribution. This is done in Figure 4. The effective hull shape lies along the streamline that starts on the horizontal axis: it closely resembles the original hull shape in Figure 2 but there are minor unimportant differences. We can conclude that this type of source distribution is a very simple and effective means of representing this type of submarine hull.

The Bernoulli Hump

If the submarine generates a disturbance in the fluid that is static with respect to the submarine, its wake is also static in the submarine's reference frame. The Kelvin wake is created by the flow around the hull and, except for turbulence, falls into this category. At the fluid surface it can be divided into two parts: the first is the near-wake, which is called the "Bernoulli hump", and the far wake, which is the usual Kelvin pattern of waves.

The fluid flow near a submarine at depth can be calculated directly from the distribution of sources and sinks as has been done for Figures 1 and 4. Near the surface, account must be taken of the surface boundary conditions. The calculated estimates of the surface displacements and velocities are simplified if the equations are linearized. Hershey [1] shows that linearization introduces only small errors of a few percent in cases of practical interest. This is to be expected as the surface disturbances are typically quite small being of the order of centimeters or less and millimeters per second. Moreover, the agreement between his measured and predicted Kelvin wake results is typically within a factor of two; the discrepancies are attributed to the neglect of turbulence, which significantly alters the flows around the stern of the submarine.

The treatment below is similar to Hershey's except for the coordinate system and the adoption of the thin ship approximation. The latter is not expected to create significant errors especially when the estimates apply to generic hull shapes; it renders the treatment much simpler.

The problem is usually expressed in terms of the velocity potential, φ, which is related to the fluid velocity, **u**, by:

$$\mathbf{u} = -\nabla\varphi . \tag{6}$$

For a source of strength, S, in an infinite fluid, the velocity potential at radius r is given by:

$$\varphi = \frac{S}{4\pi r} . \tag{7}$$

Taking the gradient gives the expected radial component of the velocity:

$$u_r = \frac{S}{4\pi r^2} . \tag{8}$$

A right handed Cartesian coordinate system is used with the x-axis to the right, the y-axis into the page and the z-axis is upwards. The submarine is regarded as stationary with the water flowing past it parallel to the negative x-axis. The velocity potential must satisfy Laplace's equation within the water and two boundary conditions must be satisfied at the surface. The first boundary condition is a kinematic condition. In this case the shape of the hump is stationary in the reference frame of the submarine so that, in this frame, the component of fluid velocity normal to the surface must be zero. A full analysis of the Kelvin wake shows that the near wake is associated with the horizontal component of the flow, while the vertical component is associated with the Kelvin wake waves. Accordingly for the near wake, we consider the vertical velocity component at the surface to be zero. This condition can be satisfied if an image of the submarine is introduced above the water surface to cancel the vertical velocity. The presence of the image doubles the surface horizontal velocities.

The second condition is that the pressure on the surface is constant atmospheric pressure. Bernoulli's equation can be written in terms of the surface height perturbation, ζ, which is measured positive upwards:

$$\frac{1}{2}\left(\left(U_x - \frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right) + g\zeta = \frac{U_0^2}{2} , \tag{9}$$

where the free stream velocity, $\mathbf{U} = -U_0\mathbf{i}$, is along the negative x-axis and g is the acceleration due to gravity. On discarding all high order quantities, this becomes:

$$U_0 \frac{\partial\varphi}{\partial x} + g\zeta = 0 . \tag{10}$$

Therefore, taking into account the image by inserting a factor of 2, we have:

$$\zeta = -\frac{2U_0}{g}\frac{\partial\varphi}{\partial x} . \tag{11}$$

For a single source and its image, (7) and (11) yield:

$$\zeta = \frac{SU_0 x}{2\pi g r^3} . \tag{12}$$

If the source is at depth h, r becomes:

$$r = (x^2 + y^2 + h^2)^{1/2} . \tag{13}$$

For the source in a doublet model corresponding to a submarine of maximum radius *a*, we have:

$$\zeta = \frac{a^2 U_0^2 x}{2gr^3} = \frac{a^2 U_0^2 x}{2g(x^2 + y^2 + h^2)^{3/2}} . \tag{14}$$

Forward of the source, *x* is positive and the water surface is elevated. Between the source and sink the surface is depressed and after the sink it is elevated again. The maximum depression is often, but not always, mid-way between the source and sink. If the origin of coordinates is placed over the submarine centre and the diameter of the submarine is *D*, the net surface elevation from the source and sink is given by:

$$\zeta = \frac{D^2 U_0^2}{8g}\left(\frac{x-(L-a)/2}{((x-(L-a)/2)^2 + y^2 + h^2)^{3/2}} - \frac{x+(L-a)/2}{((x+(L-a)/2)^2 + y^2 + h^2)^{3/2}}\right). \tag{15}$$

Hershey effectively partitions this result into two. The first is a factor with the dimensions of length and the second is a dimensionless shape factor, $f(x, y, h, L)$:

$$\zeta = \frac{D^2 U_0^2}{8gh^2} f \tag{16}$$

The tables of Stefanick and Daly are based on (16) with *f* set at a constant value of 0.8, which is adopted from an example in Hershey's report. This introduces significant inaccuracy in some scenarios.

It is straightforward to calculate and plot the elevation as a function of position for an Ohio-class submarine directly from (15). The length of the submarine is 170 m and its mean diameter is roughly 12 m. Figure 5 shows the elevations for depths of 30 m and 100 m for a speed of 10 m/s.

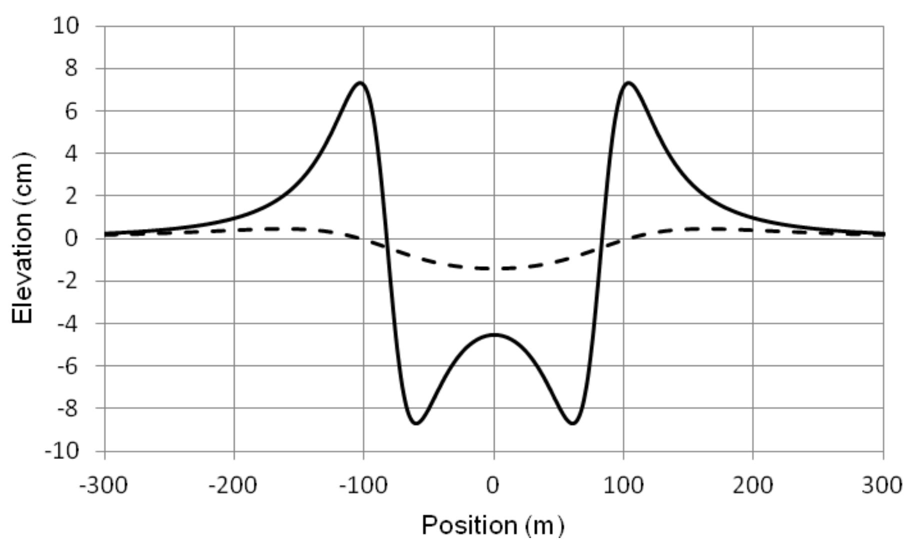

Figure 5. Profile of the Bernoulli hump directly above the Ohio-class submarine ($y = 0$) using the Rankine ovoid model. Solid line: 30 m depth. Broken line: 100 m depth.

At patrol speeds of 2.5 m/s, it is clear from (15) and Figure 5 that the surface elevations are millimeters or less.

The Bernoulli hump can be found using the new streamlined model in Figure 2 using the same method and integrating over the source distribution. The corresponding result is shown in Figure 6; the speed is again 10 m/s. The graph exhibits similar gross features but is no longer symmetrical across mid-ships.

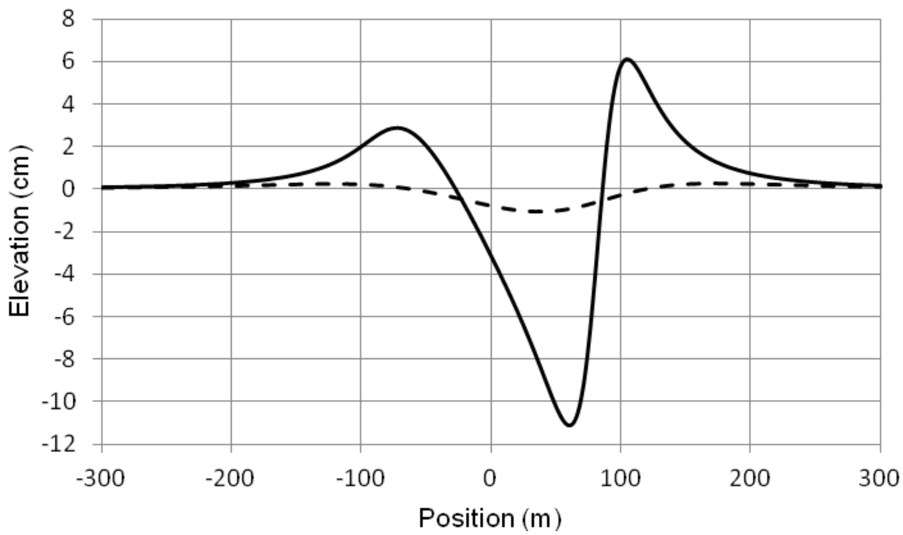

Figure 6. Profile of the Bernoulli hump directly above the Ohio-class submarine ($y = 0$) using the streamlined model. Solid line: 30 m depth. Broken line: 100 m depth.

In both Figures 5 and 6, the Bernoulli humps are better described as Bernoulli depressions. The depression directly above the submarine is to be expected because the water between the submarine and the surface is moving faster and, as is well known (for example by comparison with the Venturi effect), this tends to reduce the pressure in this region. Because the atmospheric pressure is constant on the surface, the pressure difference corresponds to a depression in the water level.

The elevations are much less than those in [2] and [3] but the peak-to-peak amplitudes of the disturbances are of a similar magnitude to the elevations quoted. Stefanick quotes an elevation of 19 cm for the Ohio-class submarine moving at 20 knots (about 10 m/s) at a depth of 30 m and 1.6 cm for the submarine at 100 m depth.

Conclusions

The tables in [2] and [3] seem to overestimate the maximum surface elevation. However, these elevations could possibly be interpreted as peak-to-peak disturbances. The maximum peak-to-peak disturbance for a large submarine can reach about 17 centimeters when it is traveling at high speeds at shallow depth; otherwise at normal patrol speeds and depths, the maximum peak-to-peak disturbance is of the order of a millimeter or less.

The Bernoulli hump is just one part of the Kelvin wake and it should be appreciated that Kelvin wake waves will be generated behind any disturbance. These will modify the profiles so that the profiles generated here are only a part of the description of the wake. A study of submarine detectability requires the complete theory.

When compared with ambient wave heights even in low sea states, it appears that the Bernoulli “hump”, which is spread out over distances of the order of 100 m, is usually difficult to detect.

References

[1] A.V. Hershey, “Measured versus computed wave trains of a Rankine ovoid”, U.S. Naval Weapons Laboratory, Dahlgren, Virginia Technical Report, AD 641888, 1966.

[2] T. Stefanick, *Strategic Antisubmarine Warfare and Naval Strategy*, D.C. Heath and Co., Lexington Books, Toronto, 1987.

[3] D.G. Daly, “A limited analysis of some non-acoustic antisubmarine warfare systems”, M.Sc. Thesis, Naval Postgraduate School, Monterey, California, March 1994.

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