Theory of the Turbulent Far-Wake

James K. E. Tunaley

***Abstract*—This report summarizes the theory of turbulence for the turbulent wake in the far field. A tensor development is adopted so that the results can be applied readily to any coordinate system, including cylindrical coordinates. Most of the results for the axisymmetric wake are not new but the formulation is useful for generalizing to other applications. However, it is shown that the angular velocity of a purely swirling wake is approximately Gaussian as a function of radius.**

***Index Terms*—Turbulence, Wake.**

I. Introduction

TURBULENCE occurs in the wake behind a ship and is important for satellite surveillance using radar because it can reveal information about the ship and its propulsion system [1]. In the case of submarine, the propeller mixes the water and, in the presence of internal layers, this mixing can collapse the wake leading to a production of internal waves [2]. These phenomena can be understood by solving the appropriate equations using either algebraic methods or by numerical simulation. In either case some basic equations are needed in various coordinate systems. The most common coordinate system is cylindrical, which directly applies to the axisymmetric wake.

The Navier-Stokes and the continuity equations are the starting points for the theory of turbulence in an incompressible fluid [3]. In the following we are concerned primarily with high Reynolds numbers and the far wake; corresponding approximations simplify the theory. For example, at high Reynolds numbers, viscosity can be ignored. The Reynolds number in a wake immediately aft of a ship of length 100 m traveling at 10 m/s in water with a kinematic viscosity of about 10^{-6} m^2/s is about 10^9. Within the wake itself, it is appropriate to define another Reynolds number involving the wake diameter and the flow mean velocity relative to the ocean; this decreases with distance astern from a typical value of 10^8. In principle, this will limit the length of the wake over which the approximations are valid.

II. Theory

The continuity equation for an incompressible fluid with velocity $\boldsymbol{u}$ is:

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

The Navier-Stokes equation for an inviscid fluid is:

$$\frac{d\boldsymbol{u}}{dt} = -\frac{\nabla p}{\rho} \tag{2}$$

where p is the (scalar) pressure and ρ is the constant density [3]. The left hand side of this is a material or convective derivative.

In this report tensor calculus is employed, e.g. [4]. The tensor form of these equations, which is valid in any system of coordinates and involves covariant derivatives, is:

$$u^k_{,k} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g}u^k)}{\partial x^k} = 0 \tag{3}$$

$$\frac{\delta u^k}{\delta t} = u^k_{,j} u^j = -\frac{g^{ki} p_{,i}}{\rho} \tag{4}$$

where the left hand side of (4) is an intrinsic derivative, which in turn may be expressed in terms of a covariant derivative, and g is the determinant of the metric tensor $\boldsymbol{g}$. In the far wake and in the absence of internal layers, it can be assumed that the pressure term in the equation of motion (4) is negligible; we can just set the intrinsic derivative to zero. Therefore the equation of motion for the fluid in the far wake assumes the very simple form:

$$\frac{\delta u^k}{\delta t} = 0 \tag{5}$$

It is customary to regard the fluid velocities as a sum of mean and fluctuating components. The mean of the fluctuations is of course zero, i.e.

$$u^k = \bar{u}^k + u'^k \tag{6}$$

Inserting this into (5) and taking the mean value yet again gives:

$$\bar{u}^k_{,j}\bar{u}^j + < u'^k_{,j} u'^j >= 0 \tag{7}$$

where both the bar and the angle brackets indicate a mean. Now the contracted covariant derivative of the kinematic stress tensor can be expressed as:

$$< u'^k u'^j >_{,j} =< u'^k_{,j} u'^j > + < u'^k u'^j_{,j} > \tag{8}$$

However, the last term is zero because it contains as a factor the divergence of the velocity (3); the wake equation becomes:

$$\bar{u}^k_{,j}\bar{u}^j = -< u'^k u'^j >_{,j} \tag{9}$$

On the left hand side we have the intrinsic derivative of the mean velocity vector and on the right we have a term analogous to the pressure term in (4); the latter is a contracted covariant derivative of the kinematic stress.

The left hand side of (9) can now be converted to a form suitable for a particular choice of coordinates; for any velocity vector, $\boldsymbol{u}$:

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J. K. E. Tunaley is with London Research and Development Corporation, Ottawa, Canada. Phone: 613-839-7943; e-mail: jtunaley@london-research-and-development.com.

$$u^k_{,j}u^j = \left(\frac{\partial u^k}{\partial x^j} + \left\{\begin{matrix} k \\ j\ s\end{matrix}\right\}u^s\right)\frac{dx^j}{dt} = \frac{du^k}{dt} + \left\{\begin{matrix} k \\ j\ s\end{matrix}\right\}u^s u^j \tag{10}$$

where {} indicates a Christoffel symbol.

Consider the case where the source of the turbulent wake is moving at speed U in the negative direction along a straight z-axis. In a frame moving with the source, $u_z \to U + u_z$; in this new frame and in the far wake where the velocities in the fluid are very small in comparison with U, other second order terms are zero or can be ignored. The partial derivative with respect to time is zero and the left hand side becomes:

$$\frac{d\bar{u}^k}{dt} = \frac{\partial \bar{u}^k}{\partial t} + \bar{u}^j \frac{\partial \bar{u}^k}{\partial x^j} \approx U\frac{\partial \bar{u}^k}{\partial z} \tag{11}$$

The right hand side of (9) can be expressed [4]:

$$< u'^i u'^j >_{,j} = \frac{\partial < u'^i u'^j >}{\partial x^j} + \left\{\begin{matrix} i \\ j\ s\end{matrix}\right\} < u'^s u'^j > + \left\{\begin{matrix} j \\ j\ s\end{matrix}\right\} < u'^s u'^i > \tag{12}$$

Consider the wake created by a source moving at velocity U along the negative z-axis in cylindrical coordinates $(r, \theta, z) = (x^1, x^2, x^3)$. The metric is given by:

$$(ds)^2 = g_{ij}dx^i dx^j = (dr)^2 + r^2(d\theta)^2 + (dz)^2 \tag{13}$$

so that the metric tensor is diagonal. The only non-zero components of the Christoffel symbol are [4]:

$$\left\{\begin{matrix} 1 \\ 2\ 2\end{matrix}\right\} = -r \qquad \left\{\begin{matrix} 2 \\ 2\ 1\end{matrix}\right\} = \left\{\begin{matrix} 2 \\ 1\ 2\end{matrix}\right\} = \frac{1}{r} \tag{14}$$

At this point we can study the axisymmetric case where the mean flow is parallel to the z-axis and there is no swirl. This simplifies the basic equations further because $\bar{u}^2 = 0$ and there is no dependence of the mean flow on the azimuthal variable, θ. Therefore the z-component ($i = 3$) of (9) becomes:

$$U\frac{\partial \bar{u}_z}{\partial z} = -\frac{\partial < u'^3 u'^j >}{\partial x^j} - \frac{< u'^1 u'^3 >}{r} \tag{15}$$

However, the second term ($j = 2$) in the sum on the right hand side is zero because there is no azimuthal variation in the mean quantities and in the far wake the third term ($j = 3$) is small and can be absorbed into the left hand side. Therefore we have the simplified form:

$$U\frac{\partial \bar{u}_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r < u'^1 u'^3 >) \tag{16}$$

Now the physical components of the velocity are related to the contravariant components by a factor $\sqrt{(g_{ii})}$, with no summation; for the first and third components this is just one and for the second, which does not occur, it is r. Therefore (16) becomes:

$$U\frac{\partial \bar{u}_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r < u'_r u'_z >) \tag{17}$$

Next we can derive a simplified equation for the purely swirling axisymmetric wake. In this case we are interested in the second (azimuthal) component of the velocity; the mean of this component is not a function of θ and in addition $\bar{u}^3 = 0$:

$$U\frac{\partial \bar{u}^2}{\partial z} = -\frac{\partial < u'^2 u'^j >}{\partial x^j} - 3\frac{< u'^2 u'^1 >}{r} \tag{18}$$

Making similar approximations as for the previous case yields:

$$U\frac{\partial \bar{u}^2}{\partial z} = -\frac{1}{r^3}\frac{\partial}{\partial r}(r^3 < u'^1 u'^2 >) \tag{19}$$

Expressing this in terms of the physical components of the velocity gives:

$$U\frac{\partial \bar{u}_\theta}{\partial z} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 < u'_r u'_\theta >) \tag{20}$$

The simplified results in (17) and (20) are identical to those stated by Reynolds [5].

In the far wake, the wake expands very slowly with time and its evolution can be described in terms of the diffusion of eddies. This suggests that the evolution of the wake resembles ordinary diffusion and the simplest approach involves a linear approximation in which the kinematic stress is proportional to velocity gradients. Thus, to a first approximation, the proportionality factor can be represented by an eddy viscosity tensor. In practice this would depend on the wake width, in the manner of Prandtl's mixing length theory [3] but, at a given cross section in the fluid, the wake dynamics are dominated by the movement of eddies. Therefore, especially in the far wake, it can be regarded as almost constant.

We expect the principal axes of an eddy viscosity tensor of an axisymmetric wake to coincide with the axes of a cylindrical coordinate system and so the eddy viscosity should be diagonal. To simplify the derivation we also assume that it is isotropic so that it can be regarded as a scalar constant, ε. Therefore, taking account of the symmetry of the stress, we have:

$$< u^i u^j > = -\varepsilon(g^{jk}u^i_{,k} + g^{ik}u^j_{,k}) \tag{21}$$

Then (9) becomes:

$$\bar{u}^i_{,j}\bar{u}^j = \varepsilon(g^{jk}\bar{u}^i_{,k} + g^{ik}\bar{u}^j_{,k})_{,j} \tag{21}$$

Noting that the covariant derivative of the metric tensor is zero and that the last term on the right involves the divergence of the velocity and is also zero, we have:

$$\bar{u}^i_{,j}\bar{u}^j = \varepsilon\, g^{jk}\bar{u}^i_{,jk} \tag{22}$$

In Cartesian coordinates, the right hand side is obviously the Laplacian of a vector.

There are two ways to proceed that lead to identical results. Either (22) can be evaluated directly or it can be expressed in terms of physical components by writing out the Laplacian of a vector in cylindrical coordinates. We choose the former route and briefly describe the latter in the Appendix.

Using the methods described in [4], it can be shown that:

$$u^i_{,jk} = \frac{\partial}{\partial x^k}\left(\frac{\partial u^i}{\partial x^j} + \begin{Bmatrix} i \\ j\ s \end{Bmatrix} u^s\right) - \begin{Bmatrix} s \\ j\ k \end{Bmatrix}\left(\frac{\partial u^i}{\partial x^s} + \begin{Bmatrix} i \\ s\ k \end{Bmatrix} u^k\right) + \begin{Bmatrix} i \\ k\ s \end{Bmatrix}\left(\frac{\partial u^s}{\partial x^j} + \begin{Bmatrix} s \\ j\ k \end{Bmatrix} u^k\right) \tag{23}$$

This can be evaluated using (14).

For example, consider the axisymmetric swirling wake. If $i = 2$, we have:

$$\begin{aligned} u^2_{,11} &= \frac{\partial^2 u^2}{\partial r^2} + \frac{2}{r}\frac{\partial u^2}{\partial r} \\ u^2_{,22} &= \frac{\partial^2 u^2}{\partial \theta^2} + \frac{2}{r}\frac{\partial u^1}{\partial \theta} + r\frac{\partial u^2}{\partial r} \\ u^2_{,33} &= \frac{\partial^2 u^2}{\partial z^2} \end{aligned} \tag{24}$$

The right hand side of (22) is found by weighting these terms by the diagonal components of the metric tensor and adding them together. The result is:

$$\begin{aligned} \bar{u}^i_{,j}\bar{u}^j / \varepsilon = & \frac{\partial^2 \bar{u}^2}{\partial r^2} + \frac{3}{r}\frac{\partial \bar{u}^2}{\partial r} \\ & + \frac{1}{r^2}\frac{\partial^2 \bar{u}^2}{\partial \theta^2} + \frac{2}{r^3}\frac{\partial \bar{u}^1}{\partial \theta} + \frac{\partial^2 \bar{u}^2}{\partial z^2} \end{aligned} \tag{25}$$

Now the three terms in the second row are either zero, because of the azimuthal symmetry, or are small. Therefore, with the usual approximations for the far wake, this can be written:

$$U\frac{\partial \bar{u}^2}{\partial z} = \varepsilon \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3 \frac{\partial \bar{u}^2}{\partial r}\right) \tag{26}$$

In a formal sense, this equation describes spherically symmetric diffusion in a four dimensional space and it can be verified by substitution that a solution is:

$$\bar{u}^2 = z^{-2}\exp\left(-\frac{Ur^2}{4\varepsilon z}\right) \tag{27}$$

Because the component u^2 is the angular velocity of the fluid in the swirling wake, this demonstrates that the angular velocity profile in the far wake is close to Gaussian. (The factor in front of the exponential, which is a function of z, is not relevant because the eddy viscosity varies along the wake.) This conclusion confirms the assumptions made in the theoretical treatment of a combined axial and swirling wake [6].

For the axisymmetric purely axial wake we need to consider the z-component of the flow (i.e. $i = 3$) and most of the Christoffel symbols in (23) are zero. Neglecting small terms as before, we find:

$$U\frac{\partial \bar{u}^3}{\partial z} = \varepsilon \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \bar{u}^3}{\partial r}\right) \tag{28}$$

This formally represents diffusion in two-dimensions. The solution of this again involves the same Gaussian factor for $\bar{u}^3$, which is just $\bar{u}_z$. As has been described in [7] and [8], this is consistent with numerical simulations and experiments though there are some deviations from the exactly Gaussian profile as perhaps expected.

III. Conclusion

A simplified equation has been derived for the turbulent wake in generalized coordinates. This has been applied to the axisymmetric wake for two canonical cases: that of the purely axial flow and that of the purely swirling flow. Previous results have been recovered.

It has also been demonstrated that the angular velocity far astern in a swirling wake should have a radial profile that is close to Gaussian.

Appendix

The Laplacian of a vector can be found in other than Cartesian coordinates through the vector identity [4]:

$$\nabla^2 \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla \times (\nabla \times \boldsymbol{A}) \tag{29}$$

The physical components of the gradient, divergence and curl of a vector $\boldsymbol{A}$ can be expressed in any curvilinear coordinate system using standard methods [4]. After some tedious calculations, it is found that for cylindrical coordinates we have:

$$\begin{aligned} (\nabla^2 \boldsymbol{A})_r = & \frac{\partial^2 A_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial \theta^2} + \frac{\partial^2 A_r}{\partial z^2} \\ & + \frac{1}{r}\frac{\partial A_r}{\partial r} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta} - \frac{A_r}{r^2} \\ (\nabla^2 \boldsymbol{A})_\theta = & \frac{\partial^2 A_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\partial^2 A_\theta}{\partial z^2} \\ & + \frac{1}{r}\frac{\partial A_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2} \\ (\nabla^2 \boldsymbol{A})_z = & \frac{\partial^2 A_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2} + \frac{\partial^2 A_z}{\partial z^2} + \frac{1}{r}\frac{\partial A_z}{\partial r} \end{aligned} \tag{30}$$

It is easy to see that (28) is recovered for the axial wake but for the swirling wake u_θ must be replaced by ωr, where ω is the angular velocity, u^2. Using the middle identity in (30), it can be confirmed that the right hand side of (25) is indeed consistent with the Laplacian of a vector.

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