The Zero-Angular-Momentum Turbulent Wake

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Abstract—**There are several canonical cases of the turbulent wake created by the movement of a body at constant velocity through a fluid. A new case is that of the zero-angular-momentum wake, which is relevant to contra-rotating propellers. This is treated using an extension of the approach used for the zero-linear-momentum wake. The theory indicates that the wake broadens with distance astern according to a power law with exponent 1/6. The fluid velocities decrease rapidly with distance with an exponent -5/6.**

Index Terms—**Contra-rotating Propeller, Turbulent Wake.**

I. Introduction

The evolution of turbulence behind a moving ship or a propeller is important for surveillance. The characteristics of a wake may provide useful information about the ship. Unfortunately the theory tends to be difficult and analytical solutions can only be found for a few simple cases [1]. Moreover, the solutions are only valid far from the source of the turbulence. For theoretical purposes, the relevant flows are axisymmetric. They are the linear-momentum-wake, in which the mean flow is along the wake axis, the swirling or angular-momentum wake (e.g. [2]), in which the mean flow is in the azimuthal direction and the zero-linear-momentum wake [3].

For example, the first two can be applied to a single propeller, which will produce both thrust and swirl. The zero-linear-momentum wake can be applied to a submarine moving at constant velocity. If there is minimal production of surface or internal waves, the drag from the hull may be compensated almost exactly by the propeller thrust; the total linear momentum in the overall wake is nearly zero.

The contra-rotating propeller comprises two screws rotating in opposite directions on one axis. The fluid is accelerated along the axis by the first screw, which also causes the fluid to swirl around it. The swirl represents a significant loss of energy with a reduction of the propeller efficiency. The second screw is located in the contracted race of the first screw and typically has a somewhat smaller diameter. The swirl from the first screw creates extra lift forces on the blades of the aft propeller and the effect is to recover a good part of the swirl energy. Therefore the net angular momentum in the wake is small.

The characteristics of a turbulent wake depend on the way the turbulence is created; the flows retain a memory of the initial conditions. For example, in a pure linear-momentum wake, which might be created by a moving jet, the wake is dominated by the constancy of linear momentum. This is because, when linear momentum is transferred to fluid at the wake edge, that fluid becomes a part of the wake itself.

The basic theory of wakes has been confirmed by laboratory experiments [4]–[6]. However, in practice the canonical turbulent wake cases are usually an incomplete description of the wake and there are some problems interpreting the details. For single screws, the wake exhibits both finite mean linear and angular momenta and it is necessary to combine the linear-momentum wake and the angular-momentum wake theories as has been done in [7]. For a contra-rotating screw, the linear-momentum wake must be combined with the zero-angular-momentum wake.

II. Theory

The theory of turbulent wakes can be based on finding a constant of the motion and then applying a dimensional analysis.

An idealized source for the linear-momentum wake is a cylindrical jet of incompressible fluid and the velocity is constant over its cross section. For the angular-momentum wake it is a rotating cylinder of fluid with a constant angular velocity. The idealized source for the zero-linear-momentum wake is a pair of concentric cylinders with fluid flowing axially within them but in opposite directions. Similarly the idealized source for the zero-angular momentum wake is a pair of concentric cylinders with fluid rotating azimuthally in opposite directions. Such a wake would be created by a pair of concentric paddles rotating in opposite directions. The angular velocities of the paddles would be such as to yield zero angular momentum across the fluid in the wake. For simplicity the angular velocities of the fluid would be constant with radius in each cylinder but with opposite signs in each of them.

The starting point for the theoretical analysis of wakes is the Navier-Stokes equation applied to a viscous Newtonian fluid. A Newtonian fluid is a fluid for which the shear stress is always proportional to the velocity gradient. To solve practical problems, this may be augmented by the conservation of mass and an equation of state but, for water wakes at normal temperatures, the equation of state is not required.

Neglecting pressure gradients, the Navier-Stokes equation, which relates the velocity of the fluid, $\boldsymbol{u}$, at position $\boldsymbol{x}$ as a function of time t, provides what is basically a momentum equation, which expressed in Cartesian coordinates and using

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the summation convention is:

$$\frac{\partial u_k}{\partial t} + u_i \frac{\partial u_k}{\partial x_i} = \nu \frac{\partial^2 u_k}{\partial x_j \partial x_j} \tag{1}$$

where ν is the kinematic viscosity.

The conservation of mass is expressed by the continuity equation for an incompressible fluid and, with no fluid sources or sinks, this just corresponds to zero divergence of the velocity:

$$\frac{\partial u_k}{\partial x_k} = 0 \tag{2}$$

The fluid velocity can be regarded as a sum of a mean component and a random component, i.e.

$$u_k = \bar{u}_k + u'_k \tag{3}$$

where the over-bar indicates a mean. According to this definition, the random component must have zero mean. Inserting the velocity given by (3) into (1) and (2) and taking the mean again yields:

$$\frac{\partial \bar{u}_k}{\partial t} + \bar{u}_i \frac{\partial \bar{u}_k}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\nu \frac{\partial \bar{u}_k}{\partial x_i} - < u'_k u'_i > \right) \tag{4}$$

where the angle brackets also indicate a mean. The last term is known as the kinematic Reynolds stress.

The ratio of the last two terms can be expressed as a Reynolds number. When the Reynolds number is very small, the last term can be neglected and the equation represents laminar flow with dissipation. When it is very large, as is invariably the case for a ship or propeller, the final term dominates; in general u'_k and u'_i are correlated.

The Reynolds stress term itself is sometimes modeled using a symmetric "eddy viscosity" tensor, $\boldsymbol{\varepsilon}$ [3]. This is related to the stress by:

$$< u'_i u'_j > = -\varepsilon_{ik} \frac{\partial \bar{u}_j}{\partial x_k} = -\boldsymbol{\varepsilon}\, \nabla \bar{\boldsymbol{u}} \tag{5}$$

In principle the eddy viscosity could vary with position but when the stress in (4) is replaced using (5), the equation structure is similar for both the laminar and turbulent flows. It is often assumed that the eddy viscosity is diagonal, isotropic and is effectively constant over a significant part of the wake.

In the following, a cylindrical coordinate system (r, θ, z) is used. The fluid velocity components are (u_r, u_θ, u_z). In a purely swirling wake $\bar{u}_r$ is very small and $\bar{u}_z$ is constant. If we transform to a coordinate frame moving with the source at velocity, U, along the z-direction, the mean wake variables do not vary in time and, for very large Reynolds numbers, (4) with (5) becomes:

$$U \frac{\partial \bar{u}_\theta}{\partial z} = \nabla \cdot (\boldsymbol{\varepsilon}\, \nabla \bar{\boldsymbol{u}}) \approx \varepsilon\, \nabla^2 \bar{\boldsymbol{u}} \tag{6}$$

In cylindrical coordinates and ignoring all terms in $\bar{u}_r$ and $\bar{u}_z$ this becomes:

$$U \frac{\partial \bar{u}_\theta}{\partial z} \approx \varepsilon \left(\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \bar{u}_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} \right) \tag{7}$$

Following the same type of approach as Birkhoff and Zarantonello [3], we introduce a second moment of angular momentum, M, given by:

$$M = 2\pi\rho U \int_0^\infty r^2 r \bar{u}_\theta r dr \tag{8}$$

Now we have for the axisymmetric mean wake:

$$\begin{aligned} \frac{\partial M}{\partial z} &= \frac{\partial}{\partial z}\left(2\pi\rho U \int r^4 \bar{u}_\theta dr \right) \\ &\approx 2\pi\rho\varepsilon \int r^4 dr \left(\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \bar{u}_\theta}{\partial r} \right) - \frac{u_\theta}{r^2} \right) \end{aligned} \tag{9}$$

Integrating the first term by parts twice gives:

$$18\pi\rho\varepsilon \int r \bar{u}_\theta \; r dr = 0 \tag{10}$$

The left hand side is zero because it is proportional to the net angular momentum in the wake. Similarly, the second term on the right hand side of (9) is zero. Therefore M is not a function of z and, under the simplifications adopted here, it is conserved along the wake.

To continue, the simplest procedure is to use dimensional analysis. From (6) and (8), the wake width, b, is a function only of $M/(\rho U)$ and z/U. We have:

$$b = \left(\frac{\beta M z}{\rho U^2} \right)^{1/6} \tag{11}$$

where β is a constant of the order of unity. In (8) we can replace the upper limit of the integral by b and then $b^5 \bar{u}_\theta$ is constant. Therefore, because b increases as $z^{1/6}$, the mean azimuthal velocities $\bar{u}_\theta$ fall off as $z^{-5/6}$.

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